Tobler's Law and Spatial Optimization

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A paper based upon this presentation will be published as: Church RL, "Tobler's law and spatial optimization: why Bakersfield?" in *International Regional Science Review*

A more complete title:

Tobler’s Law and Spatial Optimization: why Bakersfield?

or, more accurately:

Why you don’t need a link between Detroit & Bakersfield or do you?

Note: This presentation was given as the 2015 NARSC Presidential Address, Portland, OR

……preamble

- The key paper:

Tobler W (1970) A computer movie simulating urban growth in the Detroit region, *Economic Geography* 46: 234-240.

- The choice of Detroit & Bakersfield

The choice of Detroit should be obvious as Tobler's paper deals with the growth of Detroit. The choice of Bakersfield was a bit of a random pick. No reason……People in Bakersfield should not take this personally.

Outline

- A few definitions
    - Tobler's law (1970)
    - Spatial optimization
- The over-riding perspective of Operations Research in modeling spatial optimization problems (or why a Bakersfield-Detroit linkage)
    - The *p*-median problem as an example
- My posit: Tobler's law does make sense for Spatial Optimization, too
- A review of the background elements
    - The classical transportation problem (Hitchcock-Koopmans, 1941 & 1951)
    - The capacitated facility location problem (Baumol & Wolfe, 1958)
    - The general warehouse location Problem (GWLP) (Geoffrion, 1977)
- Bucking the trend
    - Applying Tobler's law in formulating the GWLP
    - An application
- Summary & Research Directions

Tobler (1970)

Computer movie simulating urban growth of Detroit, MI

- His stated objective: high success with a simple model.
- Tobler invoked: "Everything is related to everything else, but near things are more related than distant things.“
- In his description, he stated: “The specific model used is thus very parochial and ignores most of the world.”

Indeed, why would Bakersfield have much to do with the growth of Detroit?

It seems simple, let’s just ignore Bakersfield when modeling Detroit.....

Tobler's law

"Everything is related to everything else, but near things are more related than distant things.“

- Often referred to as the first law of geography
- Is it really a law?.........Some people think not
  - OK, it is isn't perfect, but neither is Newton's law of gravitational attraction
  - This is not a new question & there is a lot of discourse on this aspect
    - the interested reader is referred to a special issue of *Annals of the AAG* devoted to The First Law of Geography in 2004
- Others have offered up their versions of this law

  examples include:
    - "Ecology"
    - "Cognitive geography"
    - "Political behavior"
    - "Financial"

TBL: Tobler argued for simple, if it works. He argued for a way to reduce complexity based upon geographical proximity.

Spatial Optimization: a definition

- Spatial optimization involves identifying how land use and other activities should be arranged spatially in order to optimize efficiency or some other measure of goodness.
- Examples include:
  - Assignment and transportation problems (resource allocation across space)
  - Districting, zonation, and region delineation
  - Facility location (hubs, warehouses, fire stations, etc.)
  - Facility layout
  - Network design with or without congestion
  - Land use protection for species preservation

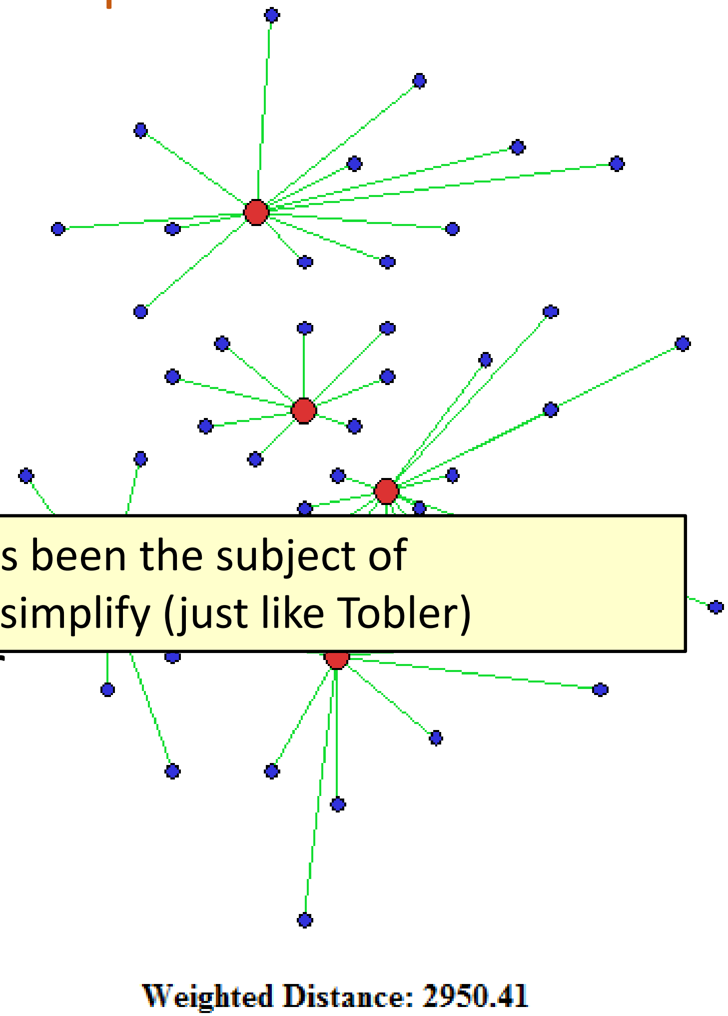
Roots of spatial optimization

- Economics: Weber 1909; Hotelling (1929); Kantorovich (1939); Koopmans (1951); Koopmans & Beckman (1957); Koopmans & Reiter(1951)
- Regional Science: Beckmann (1952); Isard (1956); Stevens (1961); Alonso (1960)
- Geography: von Thunen (1826); Christaller (1933); Garrison (1959); Alao (1970); Marble & Anderson(1972); ReVelle & Swain (1970)
- Agriculture: O'Heady and Candler (1958)
- Operations Research: Vazsonyi (Weiszfeld (1937); Dantzig, Fulkerson, & Johnson (1954); Baumol & Wolf (1958); Balinski(1964),Manne (1964), Cooper (1962), Hakimi (1964), Armour et al. (1963); Ford and Fulkerson (1962)

An example: The *p*-median problem

- Locate *p*-facilities in such a way that the average distance of serving all demand is minimized
  - Given *n* demand locations, *m* potential facility sites
  - All demand will be served by their closest facility

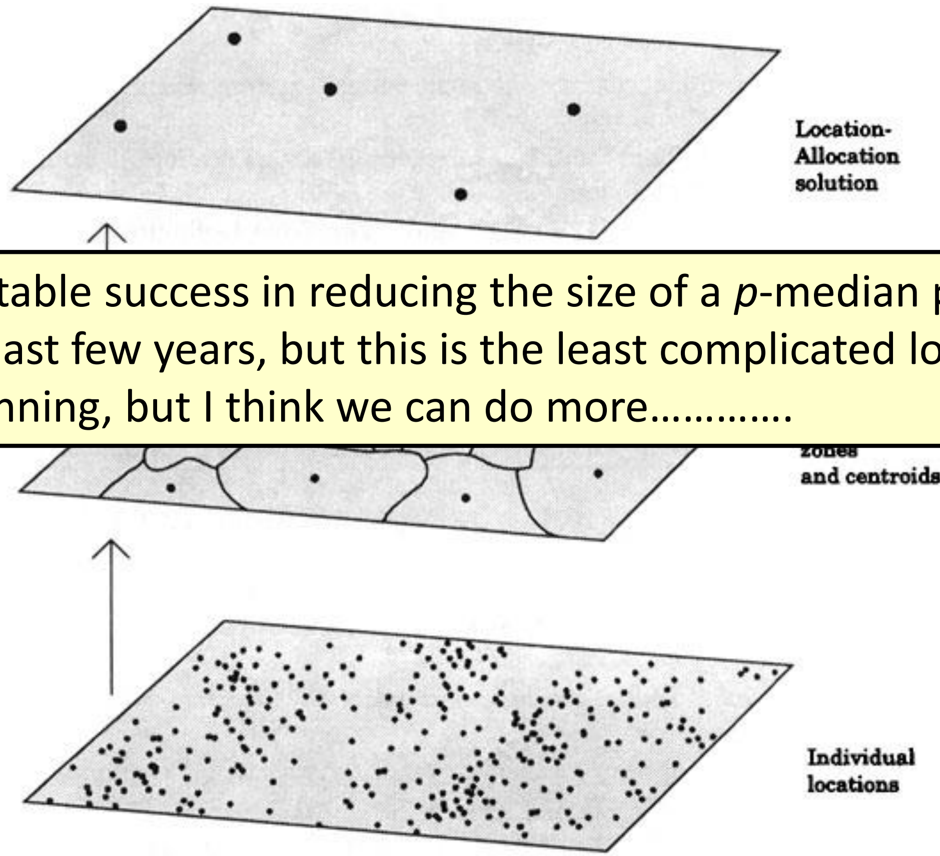
This is a classic problem introduced in the 1960's and has been the subject of considerable research including efforts to reduce it size/simplify (just like Tobler)



A 1,000 point problem with each point a feasible site as well as a place of demand has 1,000,000 variables and constraints.

Cooper (1963); Maranzana (1964); Hakimi(1965 & 1965); Teitz and Bart (1968); Vinod (1969); ReVelle and Swain (1970)

Simplifying: *the p*-median problem



There have been notable success in reducing the size of a *p*-median problem when solved to optimality in the last few years, but this is the least complicated location-allocation problem. It is a beginning, but I think we can do more…………

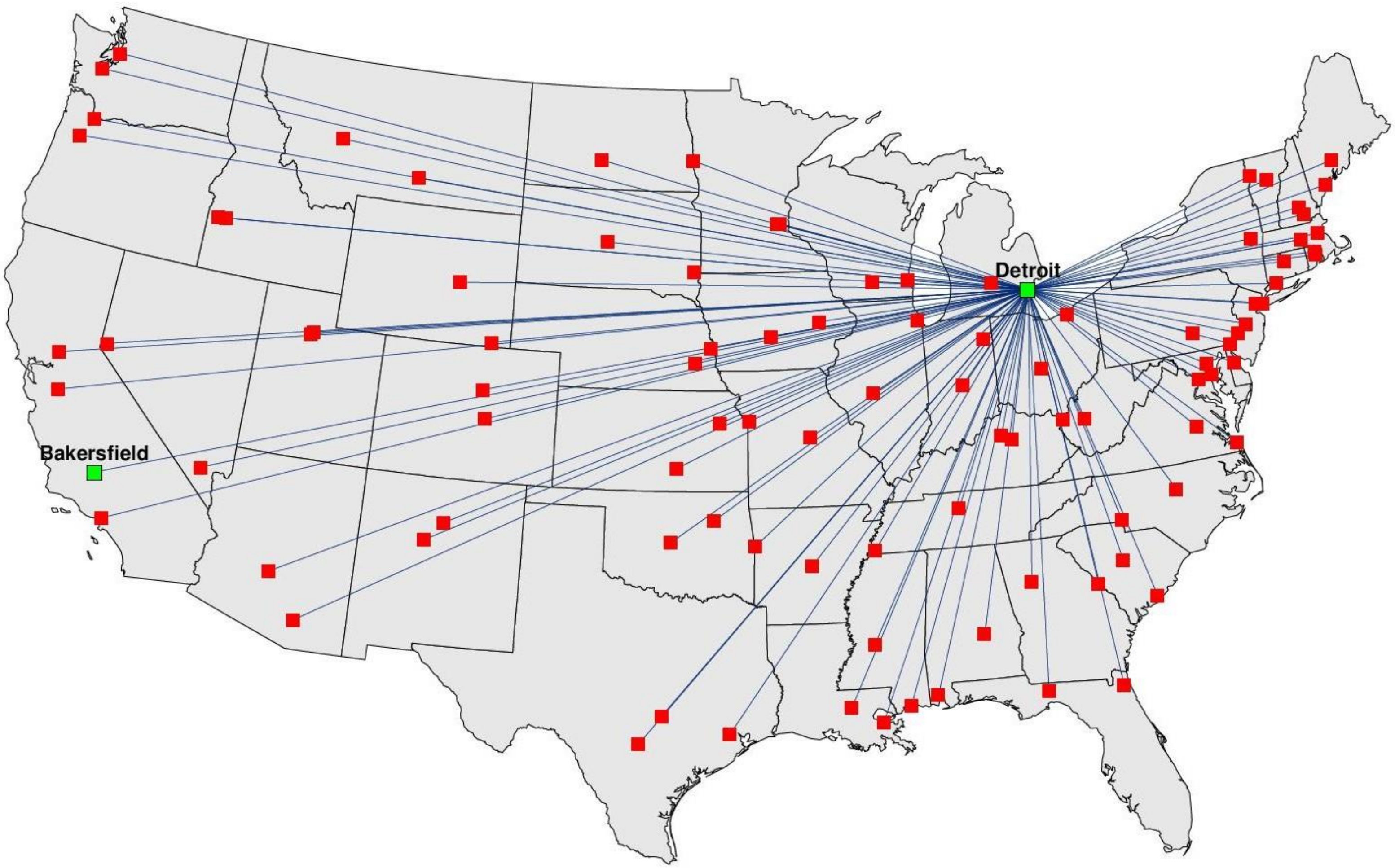
Figure from Densham et al. (1995)

OR perspective:

- Don't simplify unless you can prove it works, which often means you have to solve the complete problem to prove the case, so why bother?
- Nothing should be left to chance, models must be complete, which implies that all potential interactions are included (near and far).
- Spatial optimization models don't treat near and far differently except with respect to cost or distance of travel in an objective function or in a covering context

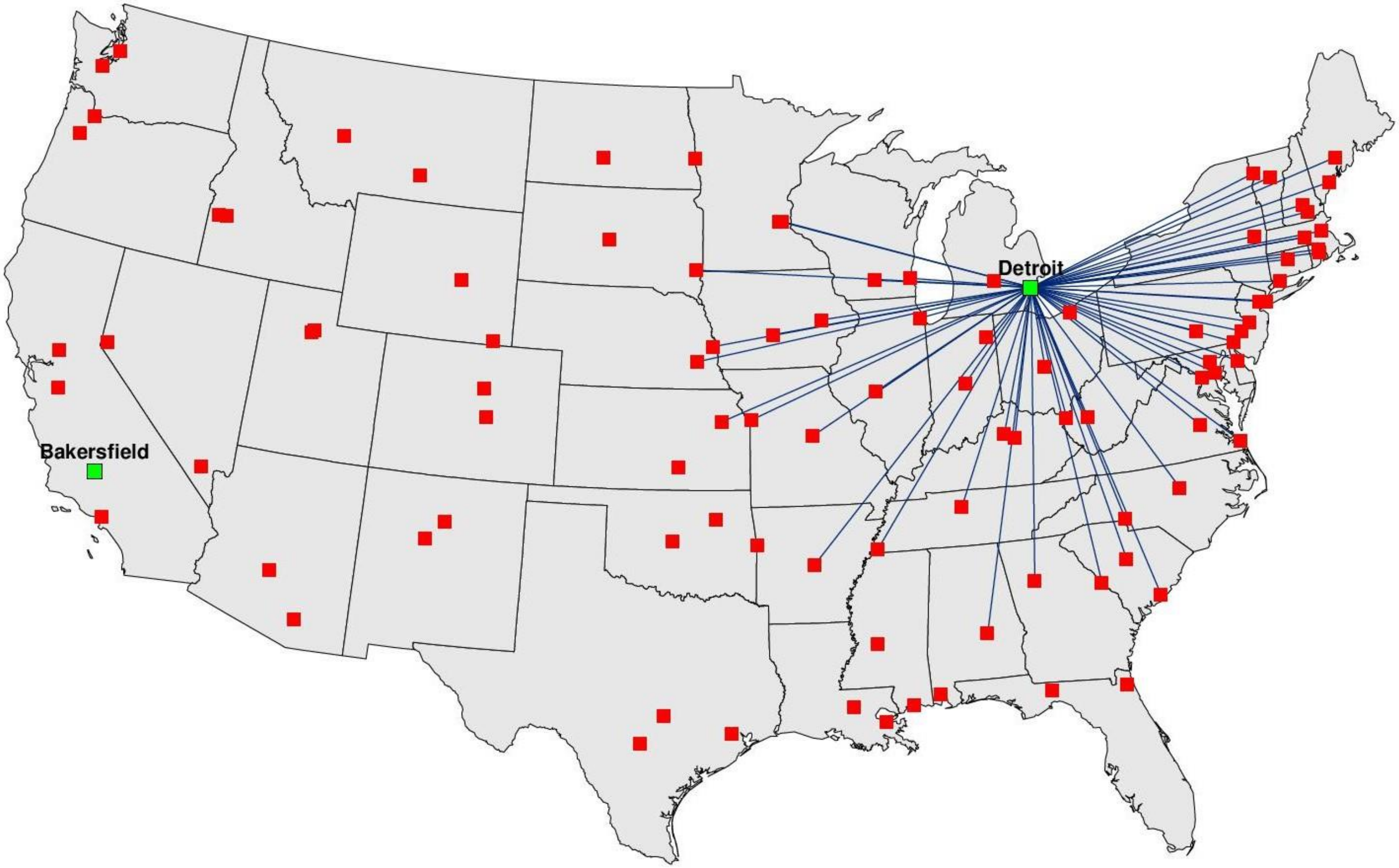
TBL: Tobler's law has had virtually no impact on what spatial opt people do, formulate, etc. In other words, we must keep all such Detroit-Bakersfield links in discrete spatial optimization problems to ensure optimality

The OR, Spatial Optimization, View……………



Detroit as a possible warehouse location in serving any one of the 100 demand areas (including Detroit itself)

The Tobler View…………..



Detroit as a possible warehouse location in serving demand areas that are within 1,000 miles: Is this enough to identify and confirm an optimal solution?

My take:

I would like to posit the following law for Spatial Optimization:

- *optimal service is more likely to be provided from a closer source than a farther one,*
- *each route of a set of optimal multiple vehicle routes is more likely to consist of a series of stops that are closer together than further apart,*
- *and so on…….*

THAT IS, I think , NEAR & FAR are concepts that we can observe in optimal solutions and we can build in our models without loss of generality

My Objective is to show how we might distinguish between NEAR and FAR assignments for the general warehouse location Problem (GWLP) (Geoffrion 1977; Beasley, 1999); that is, Tobler's law applied to a more complicated problem than the *p*-median problem…………………………….

To do this, a bit of review

The classical transportation problem (Hitchcock-Koopmans, 1941 & 1951)

The capacitated facility location problem (Balinski 1964; Manne 1964)

The general warehouse location Problem (GWLP) (Geoffrion 1977; Beasley, 1999)

Classical Transportation Problem -CTP

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to :

1) meet demand

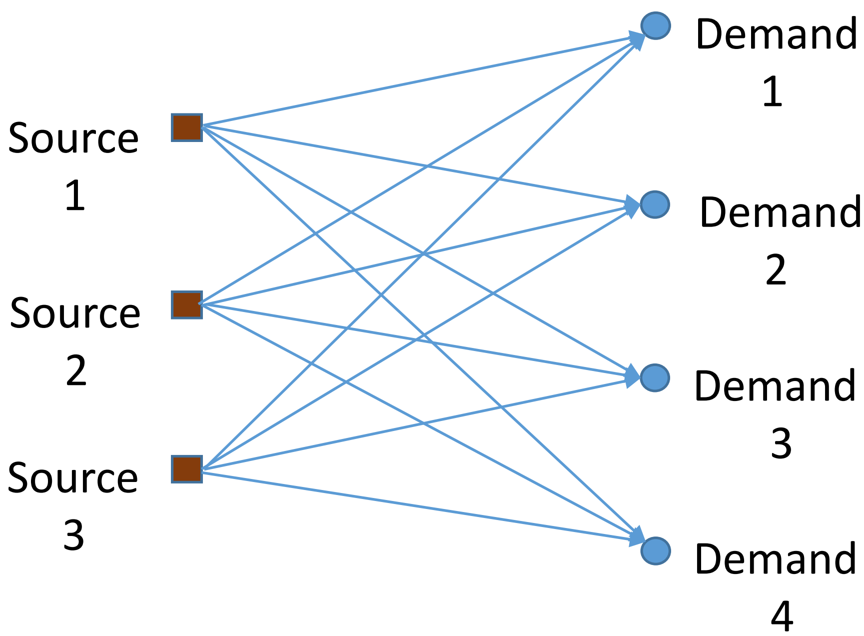
$$\sum_{i=1}^{m} x_{ij} \geq d_j \qquad \text{for each demand } j$$

2) do not overallocate supply

$$\sum_{j=1}^{n} x_{ij} \leq s_i \qquad \text{for each source } i$$

3) non - negativity constraints

$$x_{ij} \geq 0 \qquad \text{for each } i \text{ and each } j$$



x_{ij} = the number of units supplied from source i to demand j

CTP: as formulated and solved in the literature

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to :

1) meet demand

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for each demand } j$$

2) do not overallocate supply

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for each source } i$$

3) non - negativity constraints

$$x_{ij} \geq 0 \qquad \text{for each } i \text{ and each } j$$

This is based upon the assumption that :

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

x_{ij} = the number of units supplied from source i to demand j

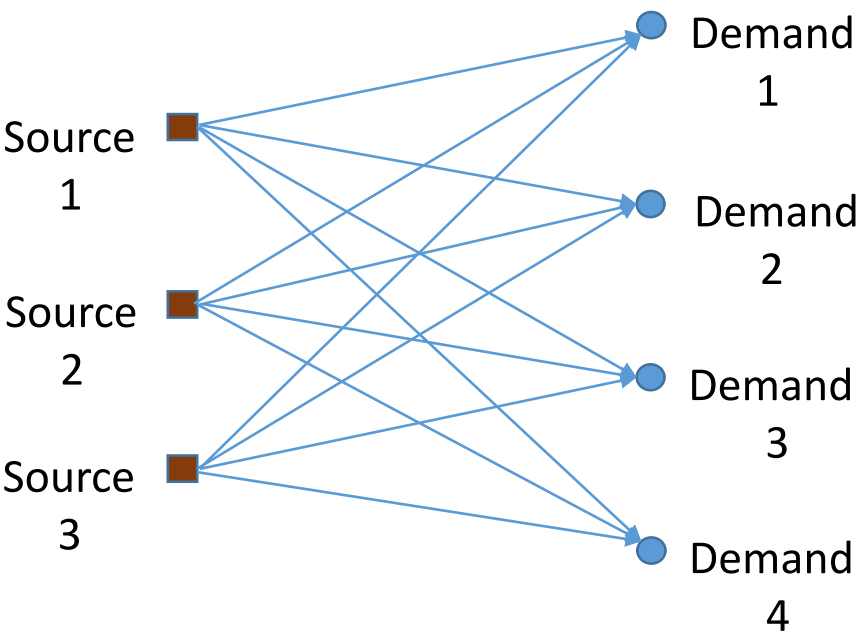
So, what do you do when $\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$

Add fictitious Demand

$$d_{n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$$

where $c_{i,n+1} = 0$ for all sources i

so, for the problem $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n+1} d_j$



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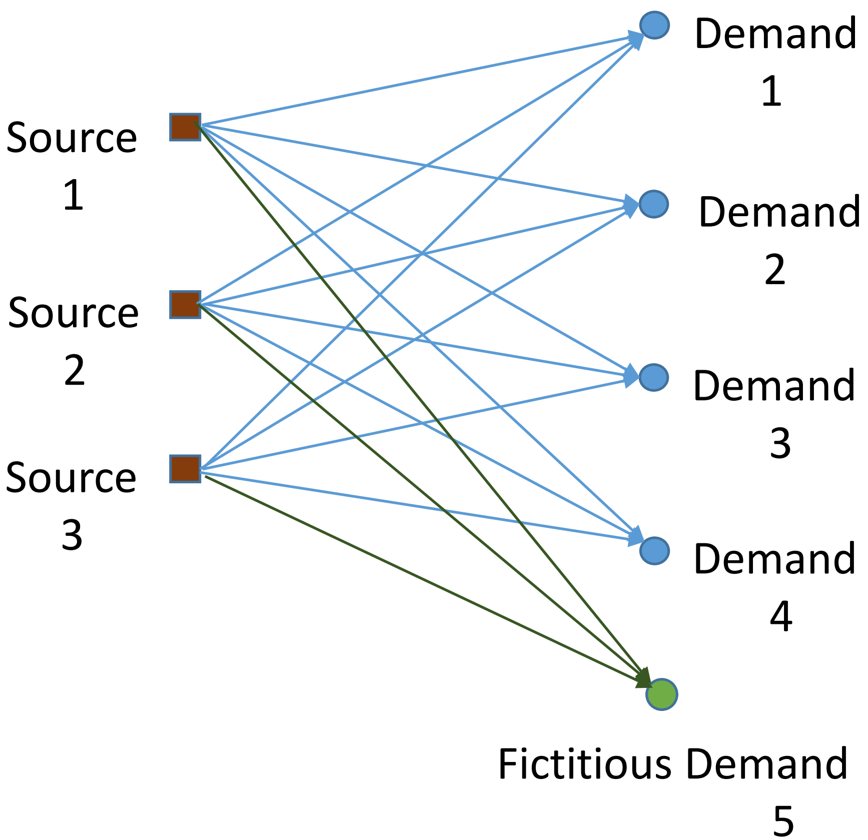


Solving the CTP

- Dantzig (1951)
- Transhipment problem of Orden (1956)
- Stepping stone method of Charnes and Cooper (1954)
  -The method that is covered in most textbooks
- Min cost flow problem - Ford Fulkerson (1956)
- Out-of-Kilter Algorithm (Fulkerson, 1961)
- Primal-Dual Algorithm Ford and Fulkerson (1957)
- Primal Flow on a pure network (Glover, Karney, Klingman, 1972)
- Primal Flow on a generalized network (Glover, Klingman, Stutz, 1973)

These last 2 approaches can be considered Fast, Fast, Fast, on very large problems

Let's Transform the CTP

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} d_j t_{ij}$

subject to :

1) meet demand

$$\sum_{i=1}^{m} d_j t_{ij} = d_j \quad \Rightarrow \sum_{i=1}^{m} t_{ij} = 1 \qquad \text{for each demand } j$$

2) do not overallocate supply

$$\sum_{j=1}^{n} d_j t_{ij} \leq s_i \qquad \text{for each source } i$$

3) non - negativity constraints

$$0 \leq t_{ij} \leq 1 \qquad \text{for each } i \text{ and each } j$$

Define $t_{ij} = \frac{x_{ij}}{d_j} =$ fraction of the demand at j that is supplied from i

A transformed CTP

t_{ij} = the fraction of the demand at j that is supplied from source i

Minimize $$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} d_j t_{ij}$$

subject to:

1) meet demand

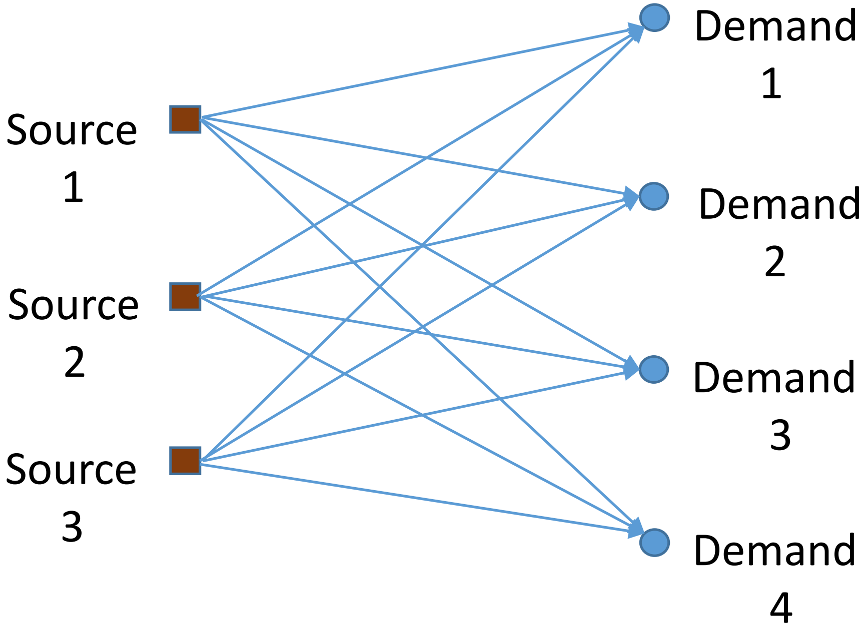
$$\sum_{i=1}^{m} t_{ij} = 1 \quad \text{for each demand } j$$

2) do not overallocate supply

$$\sum_{j=1}^{n} d_j t_{ij} \le s_i \quad \text{for each source } i$$

3) non-negativity constraints

$$0 \le t_{ij} \le 1 \quad \text{for each } i \text{ and each } j$$



Big deal, so you say…………

Locating Source Facilities:

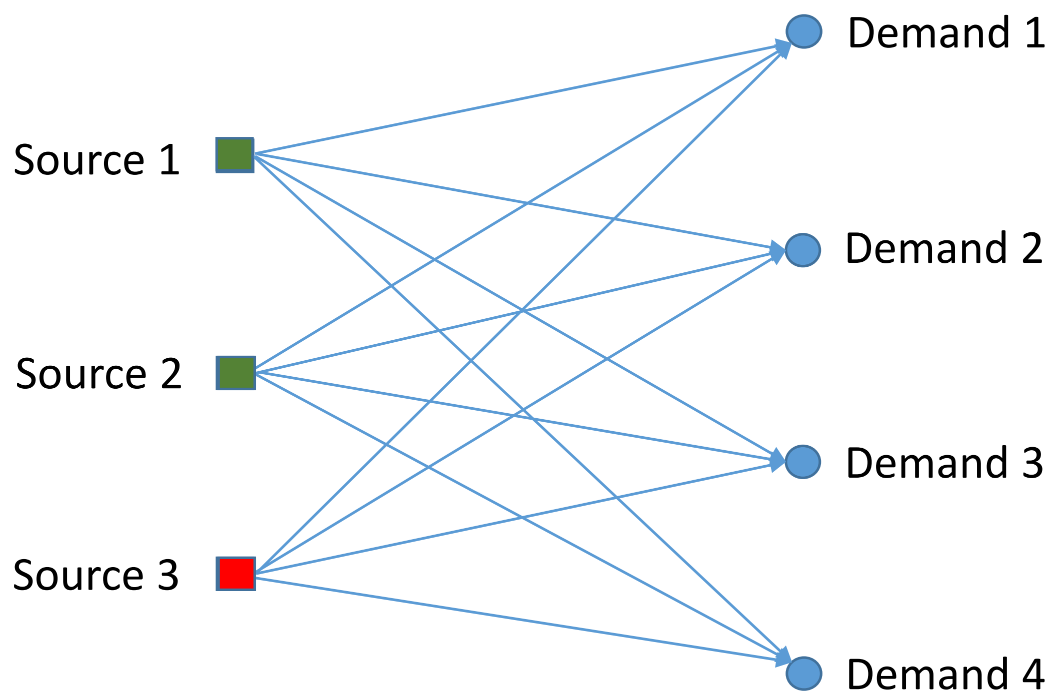
Define $y_i = \begin{cases} 1, & \text{if a warehouse is located at site } i \\ 0, & \text{otherwise} \end{cases}$

$f_i =$ fixed charge for developing a facility at site i

f_1 $\quad y_1 = 1$

f_2 $\quad y_2 = 1$

0 $\quad y_3 = 0$



The Capacitated Facility Location Problem

Incorporating the siting decision variable y_i into the CTP

$$\text{Minimize} \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} d_j t_{ij} + \boxed{\sum_{i=1}^{m} f_i y_i}$$

subject to :

1) meet demand

$$\sum_{i=1}^{m} t_{ij} = 1 \qquad \text{for each demand } j$$

2) do not overallocate supply

$$\sum_{j=1}^{n} d_j t_{ij} \le \boxed{s_i y_i} \qquad \text{for each source } i$$

3) non - negativity and integer constraints

$$0 \le t_{ij} \le 1 \qquad \text{for each } i \text{ and each } j$$

$$y_i \in \{0,1\} \qquad \text{for each } i$$

"not very integer friendly", unless we add more constraints:

$$t_{ij} \le y_i$$

The General Warehouse Location Problem of Beasley (1999)

2

Minimize $$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} d_j t_{ij} + \sum_{i=1}^{m} f_i y_i$$

subject to :

1) meet demand

$$\sum_{i=1}^{m} t_{ij} = 1 \qquad \text{for each demand } j$$

2) do not over allocate supply

$$\sum_{j=1}^{n} d_j t_{ij} \le s_i y_i \qquad \text{for each source } i$$

3) do not serve j from i unless i is a facility

$$t_{ij} \le y_i \qquad \text{for each } i \text{ and } j$$

4) locate exactly p – facilities

$$\sum_{i=1}^{m} y_i = p$$

5) non - negativity and integer constraints

$$0 \le t_{ij} \le 1 \qquad \text{for each } i \text{ and each } j \qquad y_i \in \{0,1\} \qquad \text{for each } i$$

A general form, which represents:

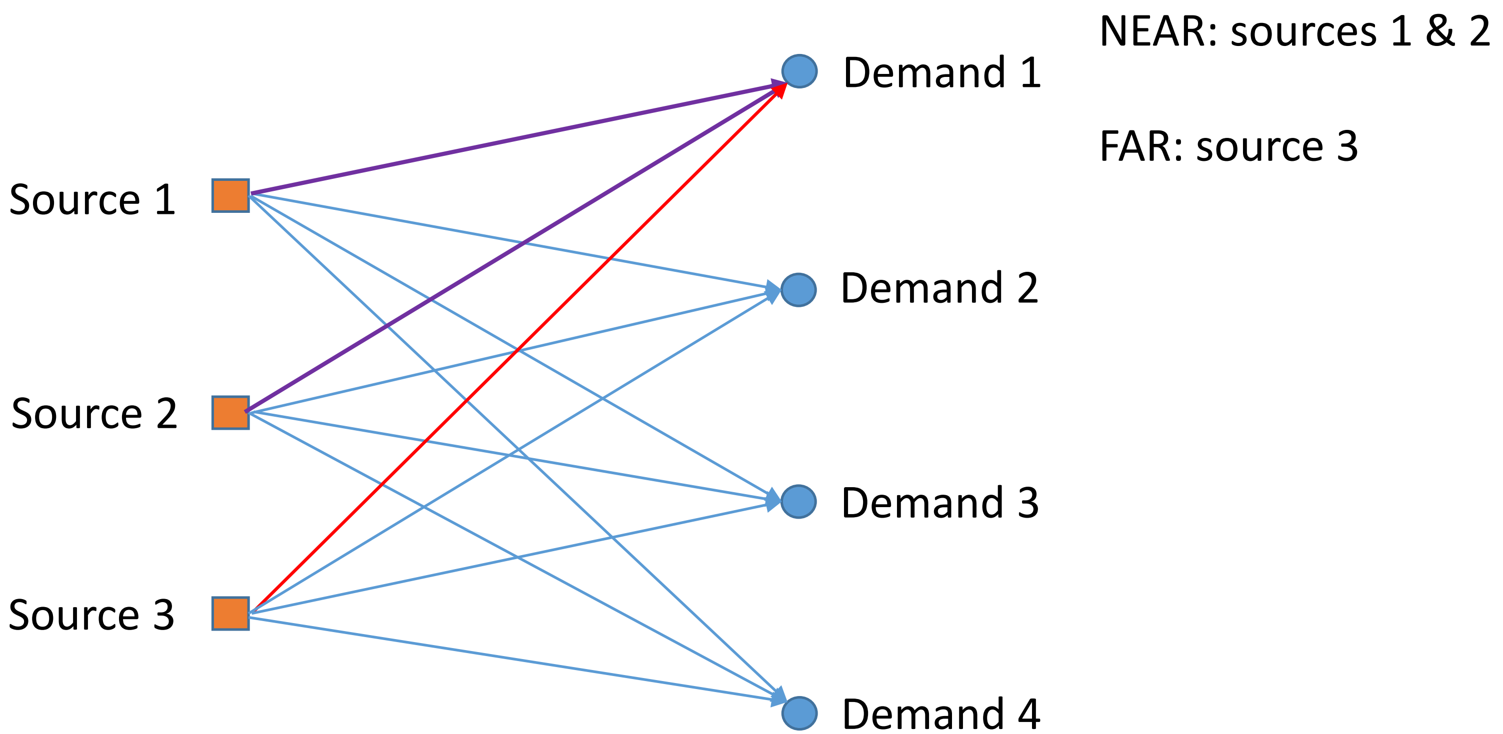
- the *p*-median problem,
- the capacitated transportation location problem,
- the simple plant location problem, and the
- capacitated fixed charge facility location problem.

But, this is a big model with a lot of variables: all linkages are represented, including Detroit to Bakersfield.

Now, what about Treating Near and Far differently?

We'll call Near: *the set of K^{th} closest facility sites for a given demand*

We'll call Far: *the set of sites that are not near, that is farther than the K^{th} closest*



Distinguish between near and far service…

Explicit service (NEAR)

r_{kj} = the fraction of the demand at j that is supplied by the k^{th} closest facility site to j

Implicit service (FAR)

G_j = the fraction of the demand at j that is supplied from a far site

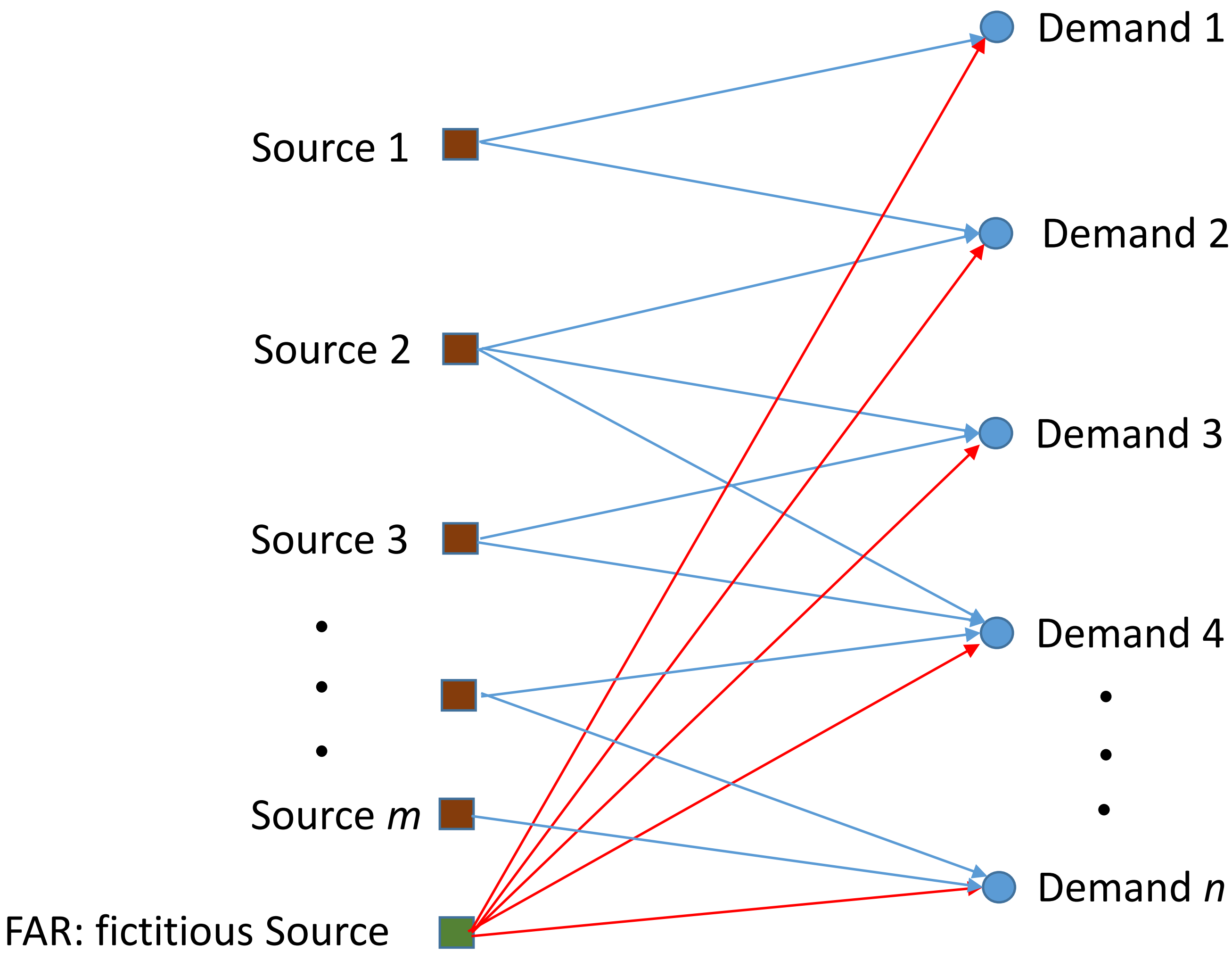
$$\sum_{i=1}^{m} t_{i,j} = 1$$

Our demand constraint can be rewritten as:

$$\sum_{k=1}^{k_j} r_{k,j} + G_j = 1$$

The basic idea is that demand *j* will be met, but beyond the K_j closest sites to demand *j*, we don't know exactly from where………

Depicting GWLP-T



We need to ensure that enough capacity is built:

$$\sum_j d_j \leq \sum_i s_i y_i$$

If enough capacity is built, then we know that FAR assignments can be honored as well as NEAR assignments

This constraint is redundant for the GWLP: all feasible configurations in the original problem must have a capacity to serve all demands

Now, The model: GWLP-T

Minimize $Z = \sum_{j=1}^{n}\sum_{k=1}^{K_j} c'_{kj} d_j r_{kj} + c'_{k_j+1,j} d_j G_j + \sum_{i=1}^{m} f_i y_i$

These are ordered costs, c', so that they match r_{kj} allocation variables

subject to :

1) meet demand

$$\sum_{k=1}^{K_j} r_{kj} + G_j = 1 \qquad \text{for each demand } j$$

Plus constraints (2) - (4)

5) $$\sum_{j=1}^{n} d_j \le \sum_{i=1}^{n} s_i y_i$$

6) modified non - negativity and integer constraints

$$0 \le r_{kj} \le 1 \qquad \text{for each } j \text{ and each } k = 1, 2, 3, \ldots, k_j$$

$$y_i \in \{0,1\} \qquad \text{for each } i$$

$$0 \le G_i \le 1 \qquad \text{for each } i$$

Note, the cost of serving demand from some FAR site is equal to transporting at the cost of the closest FAR site to *j* (i.e. $k_j + 1$)

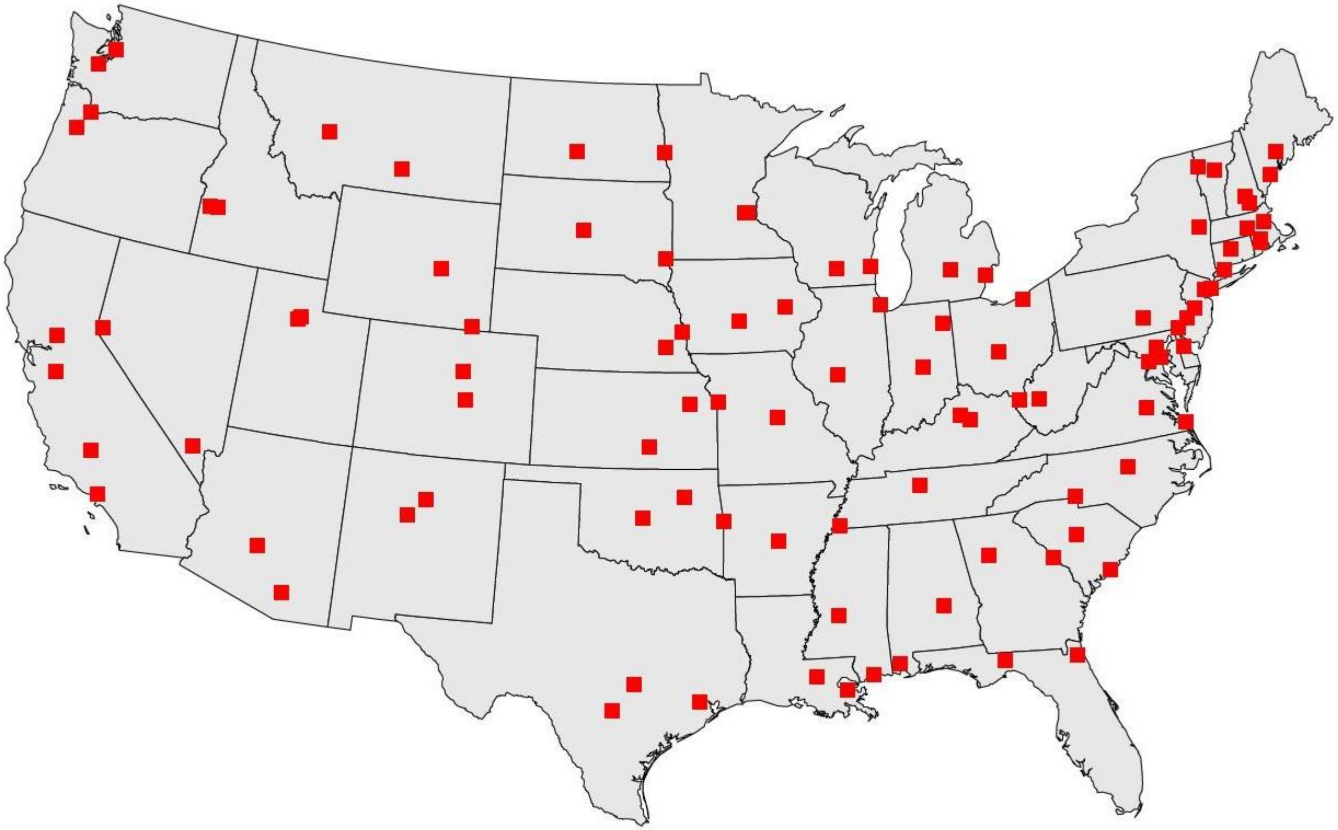
The bottom line:

- Optimal objective $Z_{GWLP\text{-}T}$ ≤ Optimal objective Z_{GWLP}
- That is, model GWLP-T provides a lower bound on GWLP
    - The proof of this is quite simple & is in the paper
- When no G_j variables appear in the solution, the two objectives are equal and the solution to GWLP-T is feasible to GWLP s and equal to the lower bound, so must be optimal, EVEN WHEN SOLVING A SMALLER, INCOMPLETE MODEL

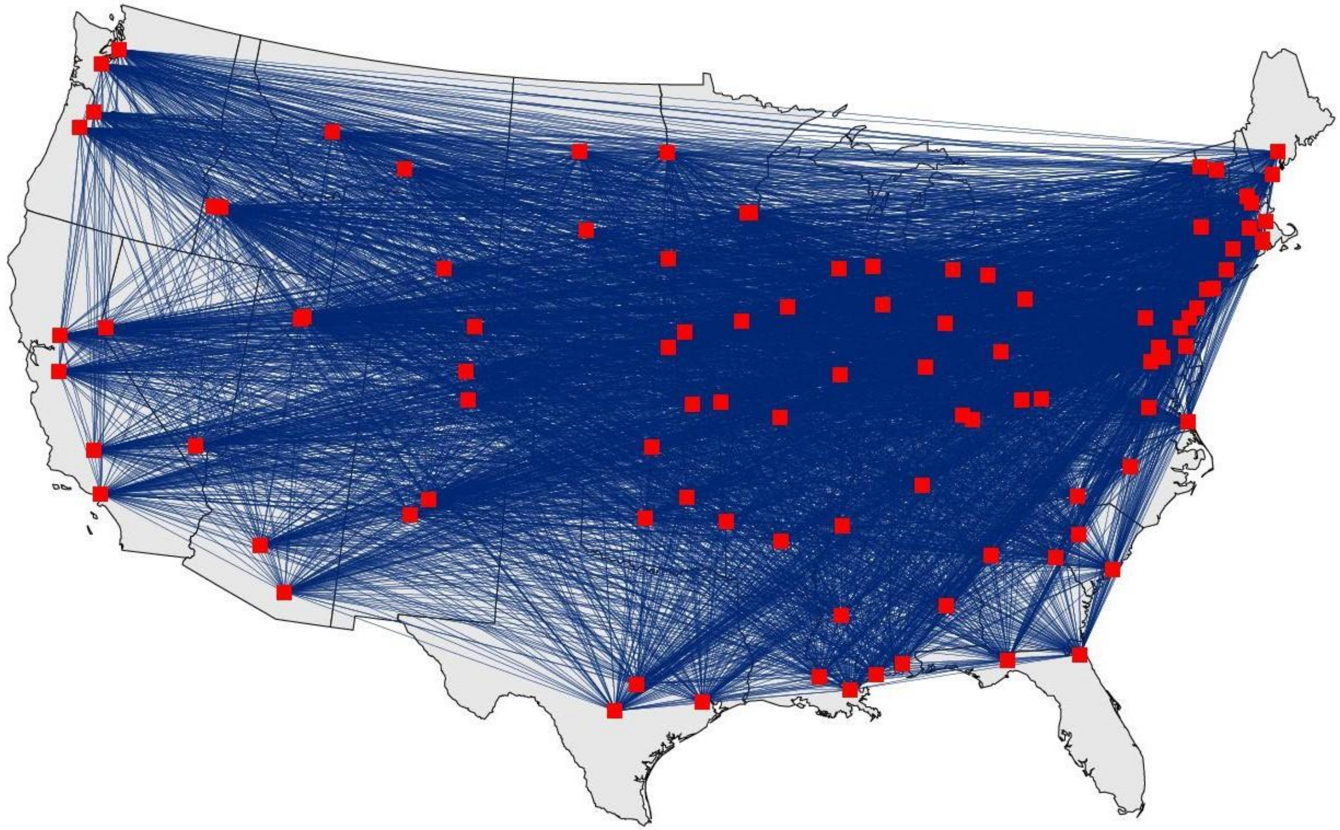
So, how do you do that?

- Step 1: start with an estimate of K_j values
- Step 2: Set up and solve GWLP-T
  - Are any G_j variables positive? If so expand the NEAR set for those demands and go back to step 1
  - Are all G_j variables equal to zero? If so stop, solution to GWLP-T is optimal to GWLP

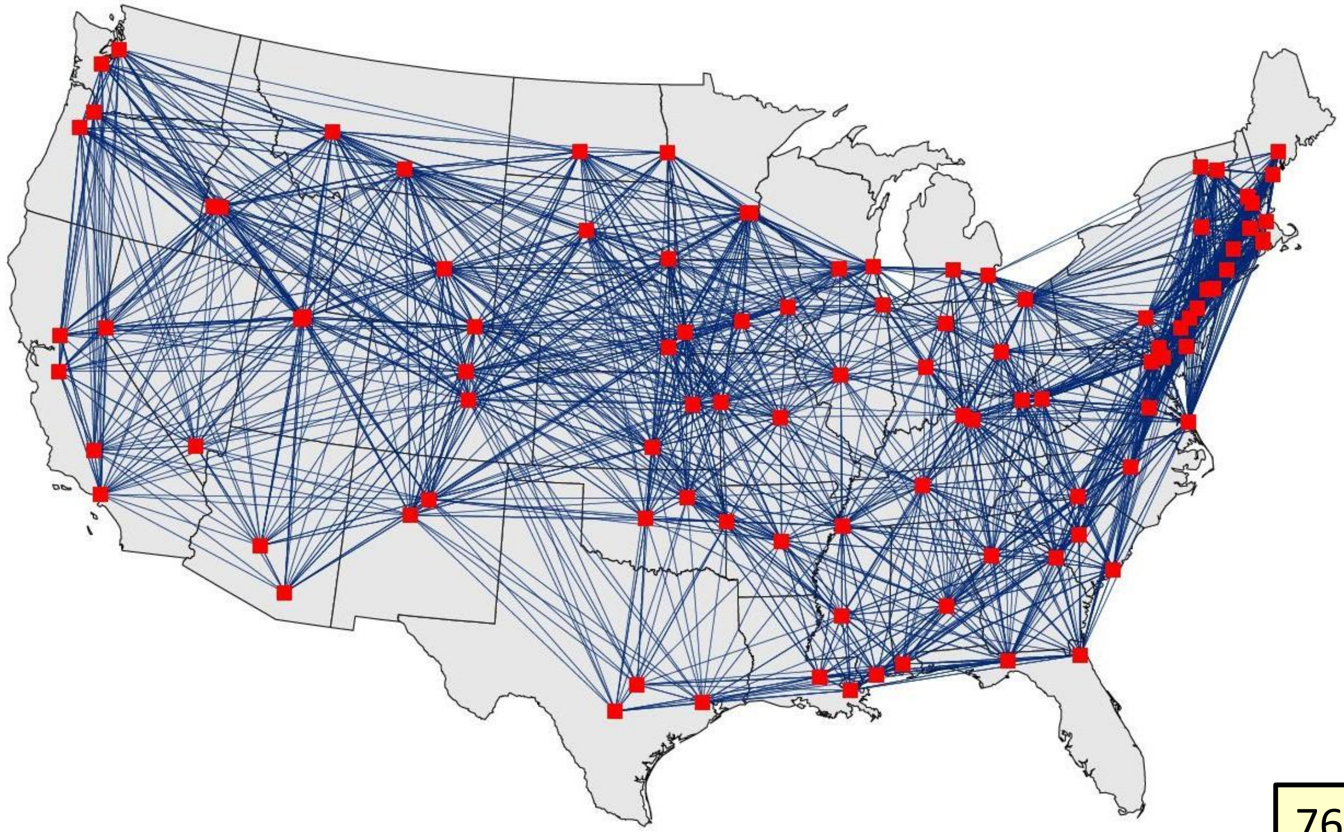
Next a few results…….



100 city data set: each city is a potential warehouse location as well as a demand point that needs to be served

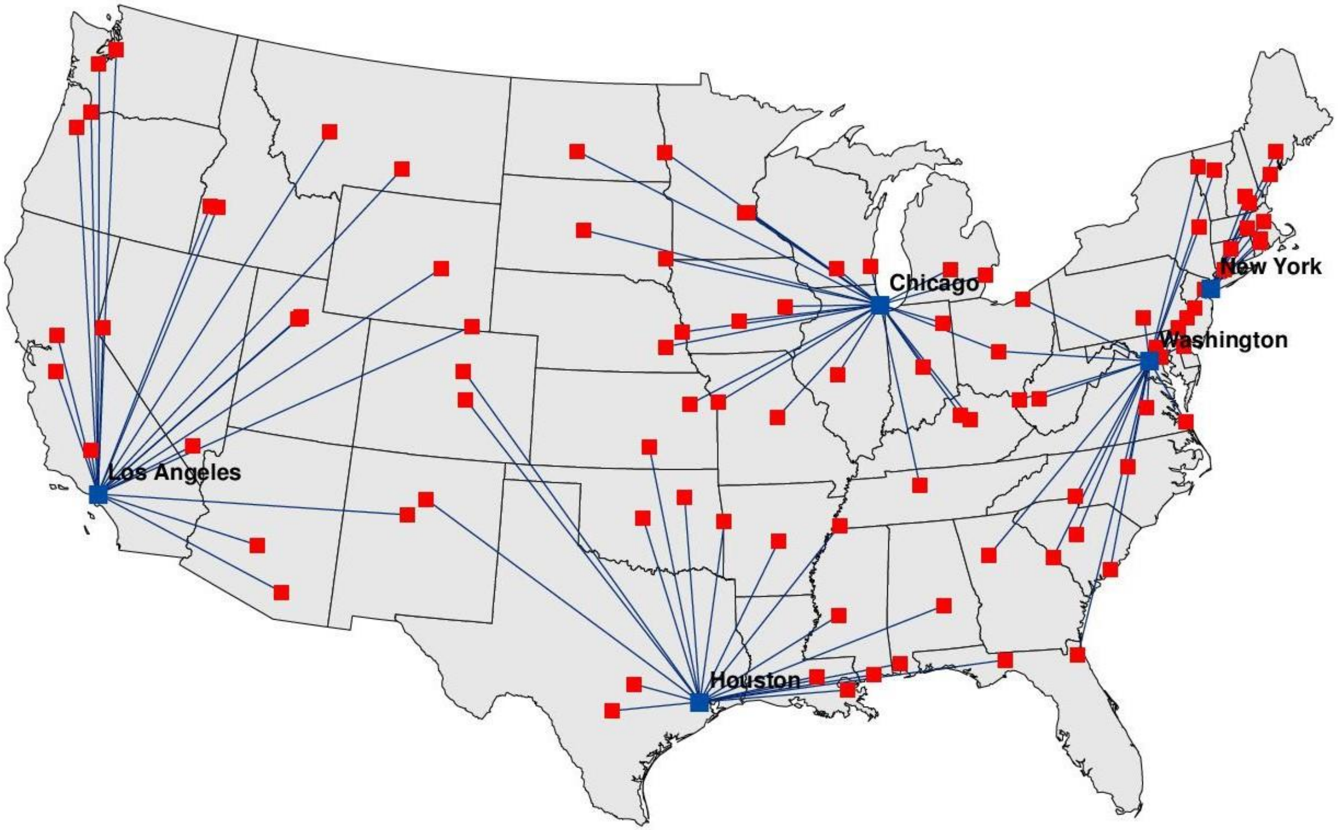


100 city data set: all 10,000 links between possible sites and demands

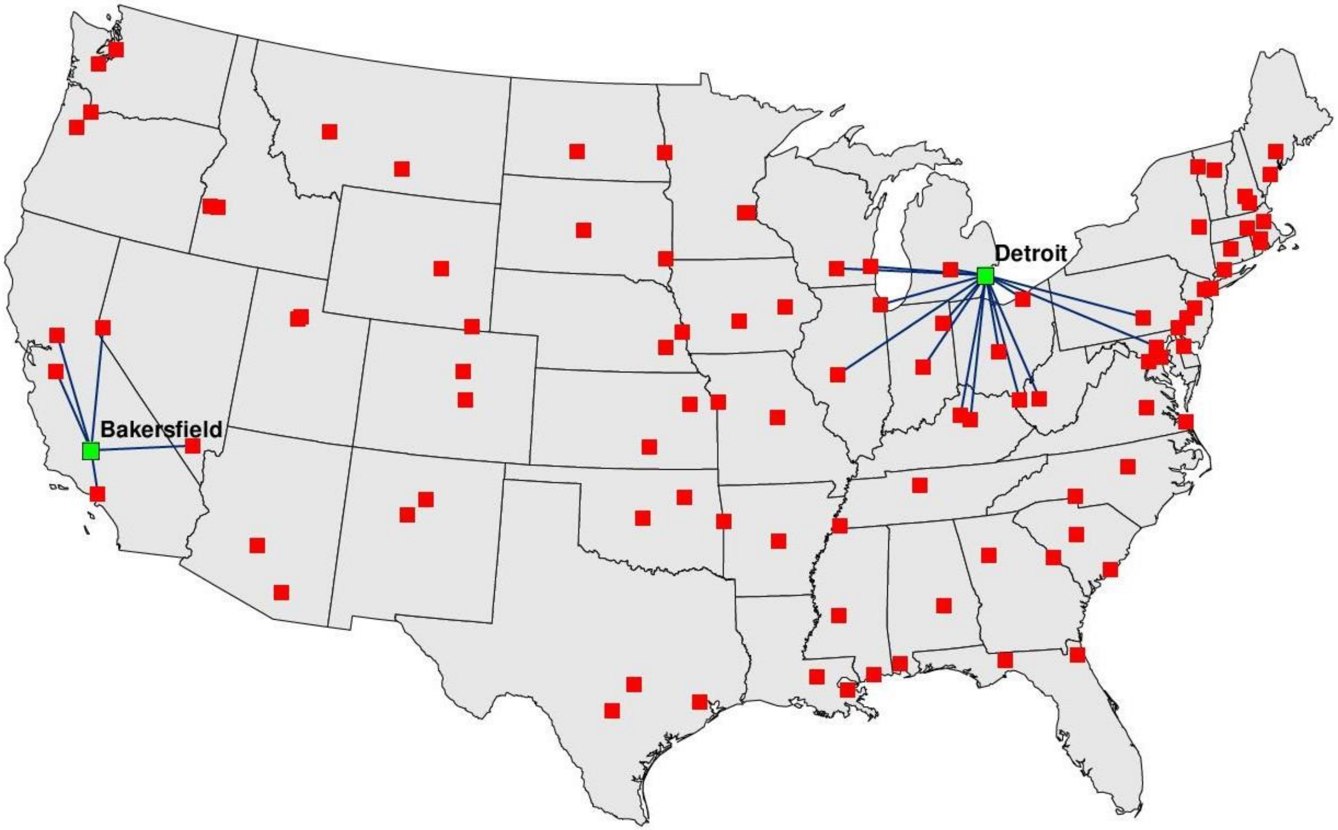


76% reduction in variables!

To find and confirm optimality for the 5 warehouse problem, only 2, 215 links were necessary



The optimal 5 warehouse solution: capacities and costs were the same for each of the possible 100 warehouse locations; supply assignments are depicted.



Links needed for Bakersfield and Detroit in finding the optimal 5 warehouse solution………

Looks like Spatial Optimization doesn't necessarily need a Detroit-Bakersfield link as well!

Table 1: computational results when solving the GWLP-T model applied to the 100 city problem					
Problem number	Facility capacities	Value of p	Sum of values at optimality	Percentage reduction	value for Bakersfield
1	10,000	5	2,215	76%	5
2	10,000	6	2,075	78%	5
3	K_{jj} 10,000	7	2,005	78%	5
4	12,000	4	3,380	65%	10
5	12,000	5	2,100	78%	5
6	12,000	6	2,065	78%	5
7	14,000	4	2,330	75%	5
8	14,000	5	1,845	80%	5
9	14,000	6	1,860	80%	5
10	16,000	3	4,790	51%	15
11	16,000	4	2,070	78%	5
12	16,000	5	1,845	80%	5
13	18,000	3	2,965	69%	10
14	18,000	4	2,070	78%	5
15	18,000	5	1,845	80%	5
16	24,000	2	5,075	48%	25
17	24,000	3	2,965	69%	10
18	24,000	4	2,070	78%	5
Total demand was 45,916.1; Initial values were set at 5 and the increment was set at 5.					

Summary

- Tobler's conceptual law can be used to help us in casting efficient approaches for many spatial optimization problems:
  - *P*-median problem
  - Fixed charge plant location
  - Capacitated Plant Location problem
- Smaller models mean larger problems can be solved, a real benefit.
- Heuristics could play a major role in determining starting values for k_j

The challenge(s)

- Specifically, What about salesman tour/cover problems, more complex distribution systems, *p*-regions problem, *p*-compact regions problem, max *p*-regions, etc.
  - All of these classic problems found in the Regional Science literature may well be amenable to new formulations based upon a spatial differences of NEAR and FAR, without compromising the search for an optimal solution.
- Generally, are there concepts of Regional Science that have been used in one area and not in another which can have broad impact on what we do; improve the efficacy of our approaches, and help to be a more integrated science?

Acknowledgements

- Waldo Tobler, for in inspiration
- Yating Chen for city maps with linkages
- FICO Corporation for access to their software