The Traffic Network Equilibrium Model: Its History and Relationship to the Kuhn-Tucker Conditions

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54th Annual North American Meetings
Regional Science Association International
Savannah, November 8, 2007

STUDIES IN THE ECONOMICS OF TRANSPORTATION

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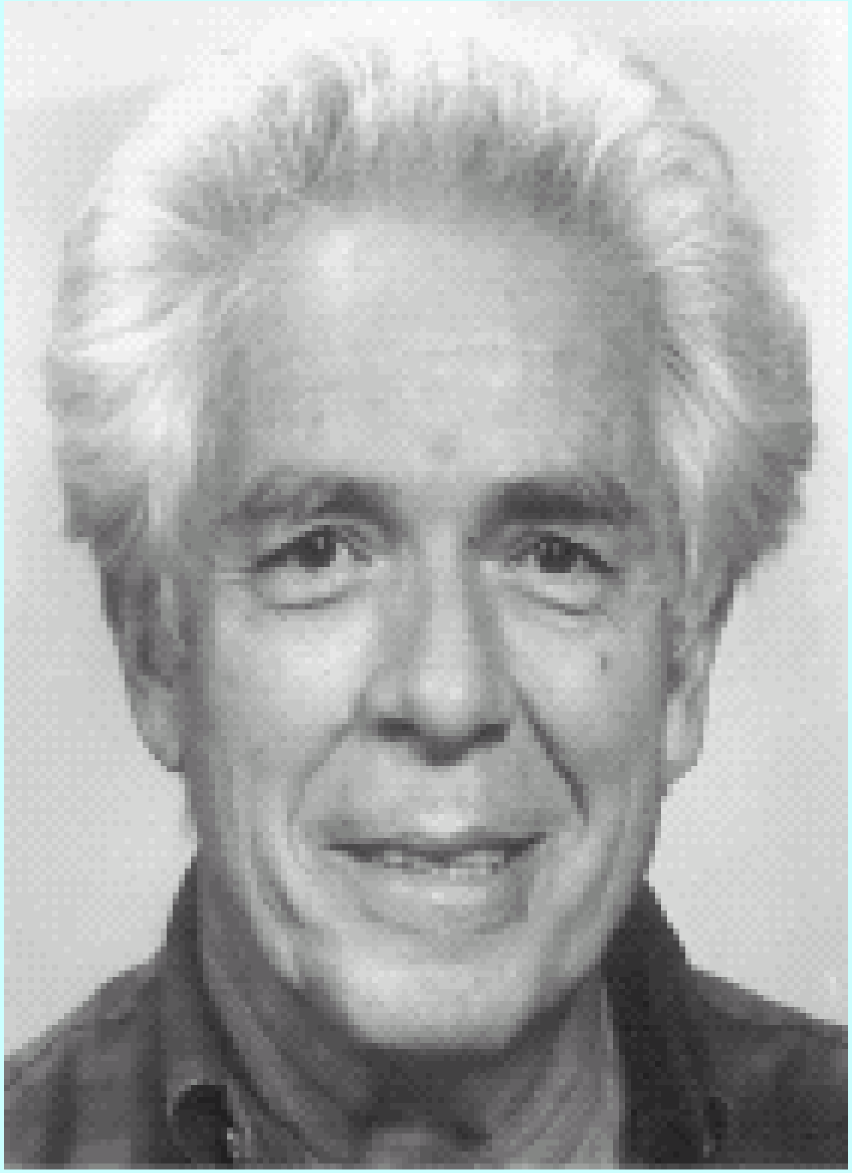
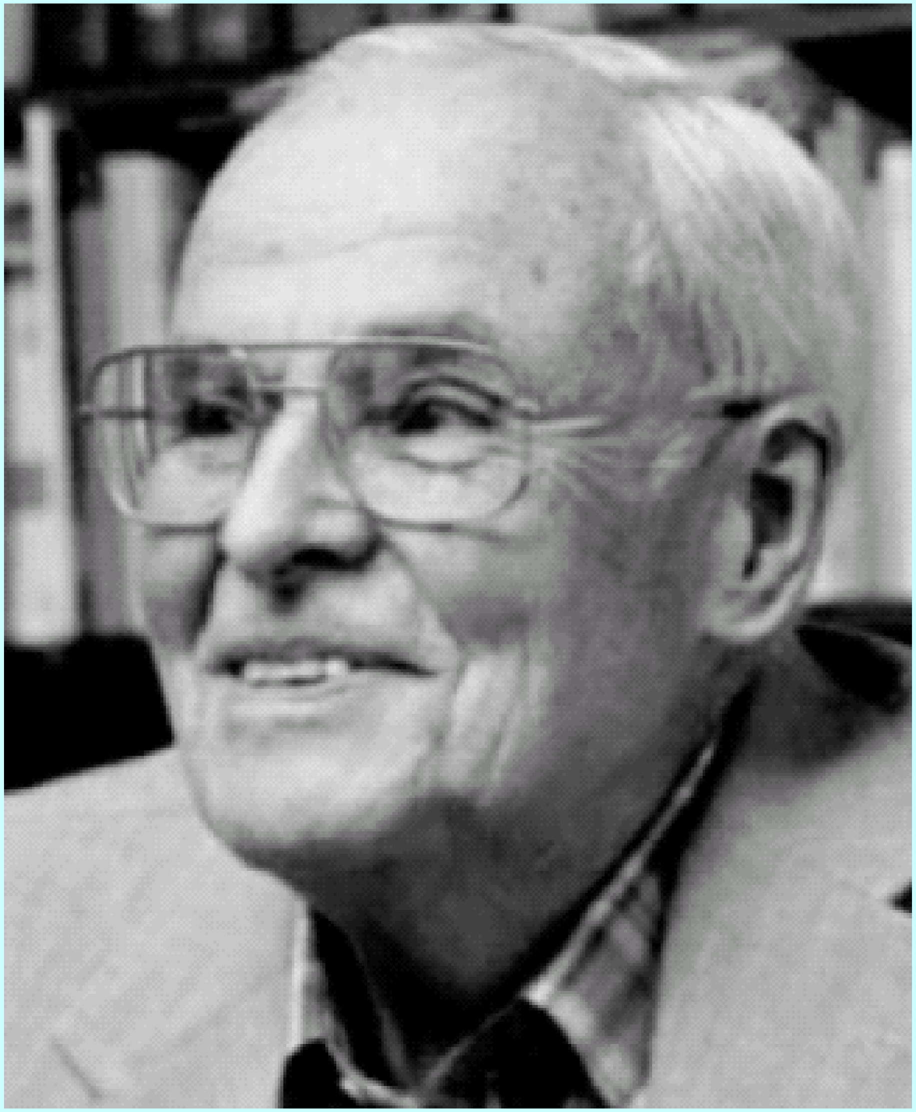
Published for the COWLES FOUNDATION
for Research in Economics at
Yale University

Dramatis Personae, 1950-1951

- Harold Kuhn, 1925 –   , born in California
  - Ph.D. in Mathematics, Princeton University, 1950
  - Lecturer in Mathematics, Princeton University, 1950-1952
  - Professor of Mathematics, Princeton University, 1959-1995
- Albert Tucker, 1905 – 1995, born in Ontario, Canada
  - Ph.D. in Mathematics, Princeton University, 1932
  - Professor of Mathematics, Princeton University, 1946-1974



Harold Kuhn, 1925- , USA



Albert Tucker, 1905-1995, Canada

Importance of the Kuhn-Tucker Theorem

- Although the origins of constrained optimization are complex, extending back to the formulation of Lagrange, Kuhn and Tucker (1951) “Nonlinear Programming,” decisively introduced this technique to economics, as well as operations research and fields of engineering, following WW II when the time was ripe for such methods;
- Kjeldsen (2000) authored a fascinating historical account in *Historia Mathematica;*
- Takayama (1985) wrote in the Introduction to his text, *Mathematical Economics*: “nonlinear programming theory … is probably the most important mathematical technique in modern economic theory.”

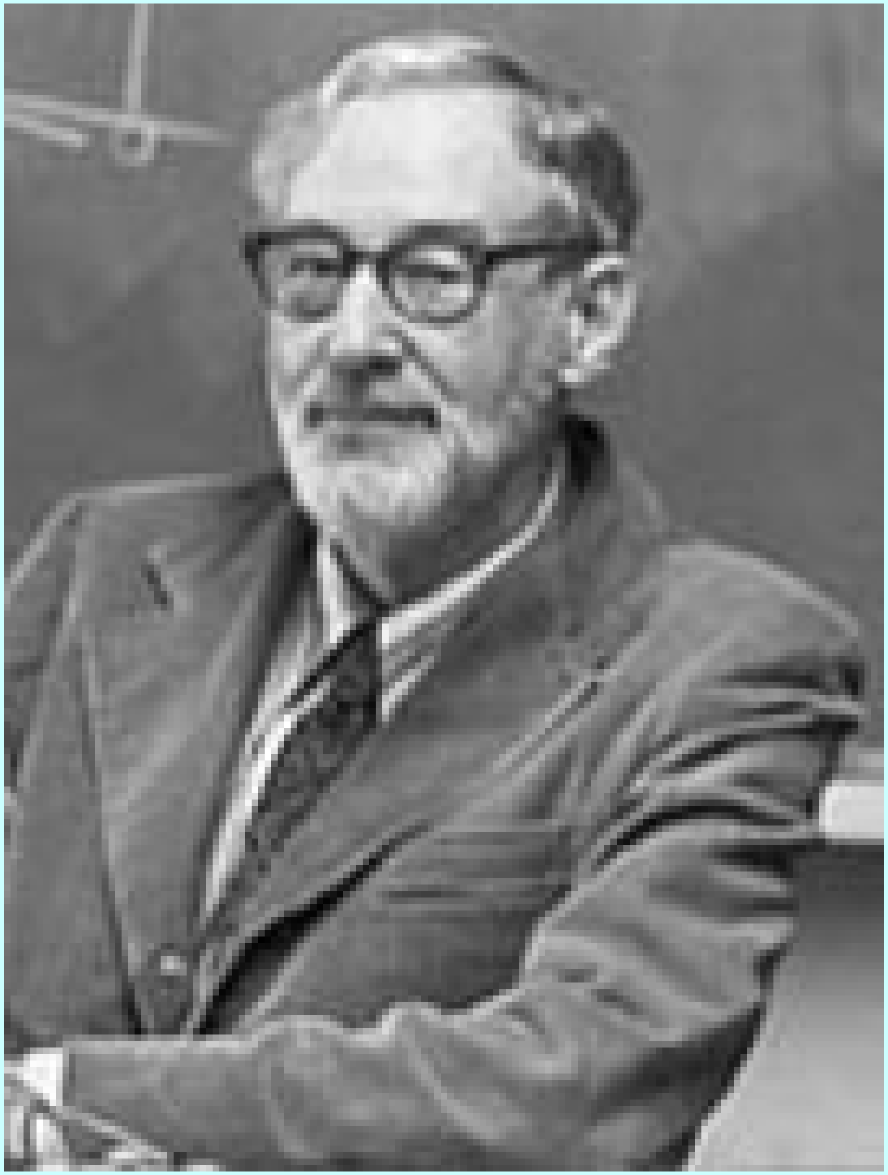
Dramatis Personae, 1951-1954

- Martin Beckmann, 1924 –   , born in Germany
  - Doctorate, Economics, University of Freiburg, Germany, 1950
  - Research Associate, Cowles Commission, 1951-1954
- C. Bartlett McGuire, 1925 – 2006, born in Minnesota
  - A. M., Economics, University of Chicago, 1952
  - Research Associate, Cowles Commission, 1952-1954
- Christopher B. Winsten, 1923 – 2005, born in England
  - B. A., University of Cambridge, UK
  - Research Associate, Cowles Commission, 1952-1954
- Tjalling C. Koopmans, 1910 – 1985, born in Netherlands
  - Doctorate, Economics, University of Leiden, Netherlands, 1936
  - Research Director, Cowles Commission for Research in Economics, and Professor, University of Chicago, 1948-1954
  - Later, Co-Recipient Nobel Prize in Economic Sciences in 1975

Martin Beckmann, 1924-   , Germany

Bartlett McGuire, 1925-2006, USA

Christopher Winsten, 1923-2005, England



Tjalling Koopmans, 1910-1985, Netherlands

A Study of the Allocation of Resources

- During 1940-54, the Cowles Commission for Research in Economics was the leading academic research center in mathematical economics and applications of mathematical programming to a broad range of problems in economics.
- Research was initiated in 1951 with support of the Rand Corporation on the “Theory of Resources Allocation,” with applications to transportation, location and population dispersal problems. Rand’s main interest was railway capacity analysis, perhaps motivated by a desire to estimate the capacity of the USSR railway system.
- The research team worked on the application of “activity analysis” to transportation and location problems. A study of efficiency in road networks led to the discovery of “traffic network equilibrium.”

Beckmann's Pioneering Achievement

- Represented a road network with general link cost functions and node conservation of flow constraints;
- Proposed a complementarity relationship for shortest route choices from an origin to a destination: if a route flow is positive, then its route cost must be a minimum, and if a route cost is not a minimum, its route flow must be zero (cf. Wardrop, 1952);
- Defined the general properties of a model of origin-destination flow (demand) as a function of endogenously determined user-equilibrium route costs;
- Formulated a concave optimization problem whose solution incorporated both demand and route choice;
- Analyzed the properties of this formulation and the related formulation for efficiency on road networks.

Significance of the Results

- Beckmann was evidently the first economist to use the Kuhn- Tucker (1951) conditions as the basis for formulating an entirely new problem in economics;
- His formulation had major practical consequences:
  - urban travel forecasting for road planning around the world;
  - road pricing policy and design of congestion toll systems;
- Beckmann may also have been the first to use the Kuhn-Tucker conditions to formulate a large-scale optimization problem in any field of engineering, making a seminal contribution to the emerging field of operations research as well as engineering in general.

Objectives of This Talk

- Offer conjectures and insights into how Beckmann achieved his result, described in Chapter 3 and 4, *Studies in the Economics of Transportation*;
- First, I review the Kuhn-Tucker (K-T) conditions;
- Then, I show how Beckmann may have applied them to formulate his model.

The Kuhn-Tucker Conditions

Find x^0 that maximizes $g(x)$ constrained by: $f_h(x), h = 1,\ldots,m$ and $x \geq 0$.

For x^0 to be a solution to the maximum problem, it is necessary that x^0 and some u^0 satisfy conditions (1) and (2), and a certain constraint qualification:

$$\left[\frac{\partial g\left(x^0\right)}{\partial x_i} + \sum_{h=1}^{m} u_h^0 \cdot \frac{\partial f_h\left(x^0\right)}{\partial x_i}\right] \leq 0,\ i = 1,\ldots n \qquad f_h\left(x^0\right) \geq 0,\ h = 1,\ldots m$$

$$(1)\ \ x_i^0 \cdot \left[\frac{\partial g\left(x^0\right)}{\partial x_i} + \sum_{h=1}^{m} u_h^0 \cdot \frac{\partial f_h\left(x^0\right)}{\partial x_i}\right] = 0,\ i = 1,\ldots n \qquad (2)\ \ u_h^0 \cdot \left[f_h\left(x^0\right)\right] = 0,\ h = 1,\ldots m$$

$$x^0 \geq 0 \qquad\qquad u^0 \geq 0$$

Setting up the Optimization Problem - 1

Let (ij) denote a directed link connecting node i to node j

$x_{ij,k}$ denotes the flow on link (ij) terminating at node k;

$x_{ij} = x_{ji} = \sum_k \left(x_{ij,k} + x_{ji,k} \right)$ denotes the total flow on link (ij)

$y_{ij} = y_{ji}$ denotes the cost of travel on link (ij) (nondirectional)

The definitions of link flow and link cost pertain to two-way roads.

Node conservation of flow (based on Kirchhoff's Law):

Let $x_{i,k} = \sum_j \left(x_{ij,k} - x_{ji,k} \right)$ be the total flow from node i to node k;

$$\sum_j x_{ij,k} = \sum_j x_{ji,k} + x_{i,k}$$

Or, flow out of node i = flow into node i + flow originating at node i

Setting up the Optimization Problem - 2

Choice of shortest routes (or least-cost route):

If $y_{i,k}$ is the least cost from node *i* to node *k*, then $y_{i,k} \le y_{ij} + y_{j,k}$, for all *j*.

For every (i,k) there is a unique value $y_{i,k}$ such that $y_{i,k} - y_{j,k} \le y_{ij}$, with the equality holding for some *j*. Assuming travelers use shortest routes:

if $x_{ij,k} > 0,$ then $y_{i,k} - y_{j,k} = y_{ij}$, and if $y_{i,k} - y_{j,k} < y_{ij},$ then $x_{ij,k} = 0,$

Therefore, $x_{ij,k}\left(y_{i,k} - y_{j,k} - y_{ij}\right) = 0$ for all nodes *j*, and all node pairs *i*,*k*.

For each pair (i,k), then, all used routes have minimal and equal costs.

Capacity or Link Performance Function

Let $y_{ij} = h_{ij}\left(x_{ij}\right)$ be the "capacity function," which is non-decreasing; where y_{ij} is the travel cost on link (ij) experienced by each traveler.

Fixed Demand Model Formulation

Consider the following cost minimization problem:

$$\max_{\left(x_{ij,k}\geq 0\right)} -\tfrac{1}{2}\sum_{ij} h_{ij}\left(x_{ij}\right)\cdot x_{ij} \qquad \text{(total cost of flow)}$$

$$\text{s.t.} \sum_{j}\left(x_{ij,k}-x_{ji,k}\right)\geq f_{i,k},\ \text{all}\left(i,k\right) \qquad \text{(node conservation of flow)}$$

where $x_{ij} \equiv \sum_{k}\left(x_{ij,k}+x_{ji,k}\right)$, for all $\left(ij\right)$,

and $f_{i,k}$ is the fixed flow from node i to node k

Define an auxiliary function, now known as the Lagrangean function:

$$L\left(x_{ij,k},\lambda_{i,k}\right)=-\tfrac{1}{2}\sum_{ij} h_{ij}\left(x_{ij}\right)\cdot x_{ij}+\sum_{i,k}\lambda_{i,k}\left[\sum_{j}\left(x_{ij,k}-x_{ji,k}\right)-f_{i,k}\right]$$

If $h'_{ij}\left(x_{ij}\right) \equiv \dfrac{\partial h_{ij}\left(x_{ij}\right)}{\partial x_{ij,k}}$, we obtain the following K-T optimality conditions:

$$\left[-h_{ij}\left(x_{ij}\right) - x_{ij} \cdot h'_{ij}\left(x_{ij}\right) + \lambda_{i,k} - \lambda_{j,k}\right] \le 0 \qquad \left[\sum_j \left(x_{ij,k} - x_{ji,k}\right) - f_{i,k}\right] \ge 0$$

(1) $x_{ij,k}\left[-\left(h_{ij}\left(x_{ij}\right) - x_{ij} \cdot h'_{ij}\left(x_{ij}\right) + \lambda_{i,k} - \lambda_{j,k}\right)\right] = 0$ (2) $\lambda_{ij,k}\left[\sum_j \left(x_{ij,k} - x_{ji,k}\right) - f_{i,k}\right] = 0$

$$x_{ij,k} \ge 0 \qquad \lambda_{ij,k} \ge 0$$

The standard K-T interpretation follows:

1) if $x_{ij,k} > 0$, then $\lambda_{i,k} - \lambda_{j,k} = h_{ij}\left(x_{ij}\right) + x_{ij} \cdot h'_{ij}\left(x_{ij}\right)$, for all (i,k)

2) if $x_{ij,k} = 0$, then $\lambda_{i,k} - \lambda_{j,k} \le h_{ij}\left(x_{ij}\right) + x_{ij} \cdot h'_{ij}\left(x_{ij}\right)$, for all (i,k)

where $\left(\lambda_{i,k} - \lambda_{j,k}\right)$ as the "cost" of travel from node *i* to node *j*.

This result shows an additional charge, or “efficiency toll,” must be placed on link (ij) for the total cost of travel to minimized. Suppose we want the link flows and costs without such tolls? This question must have confronted Beckmann. To remove the unwanted term, he altered the objective function:

$$\max_{\left(x_{ij,k}\geq 0\right)} -\tfrac{1}{2}\sum_{ij}\int_0^{x_{ij}} h_{ij}(x)dx$$

$$\text{s.t.} \sum_j \left(x_{ij,k} - x_{ji,k}\right) \geq f_{i,k}, \text{ all } (i,k)$$

$$\text{where } x_{ij} \equiv \sum_k \left(x_{ij,k} + x_{ji,k}\right), \text{ all } (i\ j)$$

Defining the objective function as the integral over the link flows yields the standard user-equilibrium result:

1) if $x_{ij,k} > 0$, then $\lambda_{i,k} - \lambda_{j,k} = h_{ij}\left(x_{ij}\right)$, for all (i,k)

2) if $x_{ij,k} = 0$, then $\lambda_{i,k} - \lambda_{j,k} \leq h_{ij}\left(x_{ij}\right)$, for all (i,k)

Variable Demand Formulation

Beckmann also assumed variable origin-destination flow (demand) depends upon shortest route costs: $x_{i,k} = f_{i,k}\left(y_{i,k}\right)$: flow from i to k is a function of the shortest route cost $y_{i,k}$ and independent of all other flows. For his formulation, Beckmann needed the inverse demand function, which he stated as: $y_{i,k} = g_{i,k}\left(x_{i,k}\right)$, assumed to be strictly decreasing

To expand the fixed demand formulation to include variable demand, we add the integral of the inverse demand function to the objective function, and change the node conservation of flows constraints to variable demand.

$$\max_{\left(x_{i,k},x_{ij,k}\geq 0\right)} \sum_{i,k} \int_0^{x_{i,k}} g_{i,k}\left(x\right)dx - \tfrac{1}{2}\sum_{ij}\int_0^{x_{ij}} h_{ij}\left(x\right)dx$$

$$\text{s.t. } \sum_j \left(x_{ij,k} - x_{ji,k}\right) \geq x_{i,k}, \text{ all } \left(i,k\right)$$

$$\text{where } x_{ij} \equiv \sum_k x_{ij,k} + x_{ji,k}, \text{ all } \left(i\,j\right)$$

The corresponding K-T optimality conditions are:

(1)

$$\left[g_{i,k}\left(x_{i,k}\right)-\lambda_{i,k}\right]\le 0$$

$$x_{i,k}\cdot\left[g_{i,k}\left(x_{i,k}\right)-\lambda_{i,k}\right]=0$$

$$\left[-h_{ij}\left(x_{ij}\right)+\lambda_{i,k}-\lambda_{j,k}\right]\le 0$$

$$x_{ij,k}\cdot\left[-\left(h_{ij}\left(x_{ij}\right)+\lambda_{i,k}-\lambda_{j,k}\right)\right]=0$$

$$x_{i,k}\ge 0;\ x_{ij,k}\ge 0$$

(2)

$$\left[\sum_j\left(x_{ij,k}-x_{ji,k}\right)-f_{i,k}\right]\ge 0$$

$$\lambda_{ij,k}\cdot\left[\sum_j\left(x_{ij,k}-x_{ji,k}\right)-f_{i,k}\right]=0$$

$$\lambda_{ij,k}\ge 0$$

From these K-T conditions, we have the following interpretations:

1) if $x_{i,k}>0$, then $\lambda_{i,k}=g_{i,k}\left(x_{i,k}\right)=y_{i,k}$, for all (i,k),

2) if $x_{i,k}=0$, then $\lambda_{i,k}\ge g_{i,k}\left(x_{i,k}\right)=y_{i,k}$, for all (i,k);

3) if $x_{ij,k}>0$, then $g_{i,k}\left(x_{i,k}\right)-g_{j,k}\left(x_{j,k}\right)=h_{ij}\left(x_{ij}\right)$, for all (ij,k)

4) if $x_{ij,k}=0$, then $g_{i,k}\left(x_{i,k}\right)-g_{j,k}\left(x_{j,k}\right)\le h_{ij}\left(x_{ij}\right)$, for all (ij,k)

Conclusions

- The Kuhn-Tucker conditions were the basis for Beckmann's formulation, including the representation of the user-equilibrium route choice conditions. The only prior application of the K-T conditions in economics found so far was by Dorfman (1951), who proved a known result in a new way.
- One key to Beckmann's formulation was representing the shortest route assumption in the optimality conditions as a complementarity relationship. Then, he formulated an equivalent, or artificial, objective function that generated the desired results.

- Later formulations of the problem changed to the link-route representation of conservation of flow (Patriksson, 1994). Although this representation may be more intuitive, some insights may be lost.
- Similar formations involving the integral of a function are found in economics, and perhaps in theoretical mechanics, as Beckmann hinted. Consumers' surplus, discussed in Chapter 4, could be viewed in the same way, although a plausible interpretation is available.
- In such cases an interpretation of the objective function may not be possible, or it may need to be interpreted in a more complex framework. An example is the "representative traveler" interpretation proposed by Oppenheim (1995).